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Bose-Einstein condensates in optical lattices: Spontaneous emission in the presence of photonic band gaps

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Abstract. An extended Bose-Einstein condensate (BEC) in an optical lattice provides a kind of periodic dielectric and causes band gaps to occur in the spectrum of light propagating through it. We examine the question whether these band gaps can modify the spontaneous emission rate of atoms excited from the BEC, and whether they can lead to a self-stabilization of the BEC against spontaneous emission. We find that self-stabilization is not possible for BECs with a density in the order of 10^{14} cm^{-3}. However, the corresponding non-Markovian behavior produces significant effects in the decay of excited atoms even for a homogeneous BEC interacting with a weak laser beam. These effects are caused by the occurrence of an avoided crossing in the photon (or rather polariton) spectrum. We also predict a new channel for spontaneous decay which arises from an interference between periodically excited atoms and periodic photon modes. This new channel should also occur in ordinary periodic dielectrics.

PACS. 03.75.Fi Phase coherent atomic ensembles; quantum condensation phenomena – 32.80.-t Photon interactions with atoms – 42.70.Qs Photonic bandgap materials

1 Introduction

It is well-known that the radiation properties of atoms can dramatically be manipulated by changing the environment where atoms emit photons. For micro-cavities it has been demonstrated [1] and for periodic dielectric media predicted [2,3] that a suppression of spontaneous emission (SE) can be achieved. In the case of a micro-cavity this happens because its geometry reduces the radiation-mode density, whereas in a periodic dielectric medium SE is suppressed due to the formation of photonic band gaps (PBG).

The recent achievement of Bose-Einstein condensation in magnetic traps [4] has provided a new state of matter where all atoms share a single macroscopic quantum state. Such a state of matter offers great opportunities to explore and test new phenomena related to macroscopic quantum coherence. Recently several authors have theoretically studied spontaneous emission in a trapped BEC. In this case the continuous center-of-mass momentum distribution leads to an increase of SE [6,7]. In addition, the stimulated emission can be increased by the Bose enhancement in a BEC [8]. In the case of two BECs, interference effects can be important [9].

The present work is focused on the case of an extended BEC and was motivated by the following idea. If an extended BEC is placed in an optical lattice it will become periodic. Since such a BEC does provide a (quantum) dielectric it affects the properties of photons propagating through it and phenomena similar to PBGs do occur [10]. Since then the photon mode density around the resonance frequency is reduced one can expect that SE is suppressed by non-Markovian effects. We thus are led to the following question: can a BEC in an optical lattice stabilize itself against spontaneous emission?

A large part of this paper is devoted to the answer of this question. However, we also have studied what will happen for a homogeneous extended BEC interacting with a weak running laser beam. Surprisingly, non-Markovian effects similar to that in a PBG do occur even in this non-periodic situation. This happens because in the presence of such a BEC photons and excited atoms do form superpositions called polaritons [5]. The spectrum of these polaritons contains an avoided crossing which has a similar effect on the SE rate as a PBG.

The paper is organized as follows. In Section 2 we will present the theoretical model on which our calculations are based. The general derivation of the SE rates in laser fields will be done in Section 3. The results for the case of a BEC in a traveling wave laser beam or in a 1D optical lattice beam are discussed in Sections 4 and 5 respectively, and are summarized in Section 6. The details of the calculations are given in two Appendices.

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2 The theoretical model

We consider a BEC composed of two-level atoms which is coupled to the electromagnetic field. The interaction is described by using minimal coupling in rotating-wave approximation under neglection of the term quadratic in the electromagnetic field. The interaction Hamiltonian is then given by

$$H_{\mathrm{int}} = \int \mathrm{d}^3k\mathrm{d}^3k' \zeta_\sigma(\mathbf{k}) a_\sigma(\mathbf{k}) \Psi_{\mathrm{g}}(\mathbf{k}') \Psi_{\mathrm{e}}^\dagger(\mathbf{k}+\mathbf{k}') + \mathrm{H.c.}, \tag{1}$$

with $\zeta_\sigma(\mathbf{k}) := \omega_{\mathrm{res}} \mathbf{d} \cdot \boldsymbol{\varepsilon}_\sigma(\mathbf{k}) [\hbar/(2(2\pi)^3\varepsilon_0\omega_k)]^{1/2}$ for an electromagnetic mode with frequency $\omega_k = c|\mathbf{k}|$ and polarization vector $\boldsymbol{\varepsilon}_\sigma(\mathbf{k})$. The vector $\mathbf{d}$ denotes the atomic dipole moment and ω_{res} is the resonance frequency. The Heisenberg equations of motion for the photon annihilation operators $a_\sigma(\mathbf{k})$ and the field operators Ψ_{e} and Ψ_{g} for excited and ground-state atoms can be derived easily and are given by

$$\begin{aligned}\mathrm{i}\hbar\dot{\Psi}_{\mathrm{e}}(\mathbf{k}) &= \left\{\frac{\hbar^2\mathbf{k}^2}{2M} + \hbar\omega_{\mathrm{res}}\right\}\Psi_{\mathrm{e}}(\mathbf{k}) \\ &\quad + \int \mathrm{d}^3k' \sum_\sigma \zeta_\sigma(\mathbf{k}') a_\sigma(\mathbf{k}') \Psi_{\mathrm{g}}(\mathbf{k}-\mathbf{k}') \end{aligned}\tag{2}$$

$$\begin{aligned}\mathrm{i}\hbar\dot{\Psi}_{\mathrm{g}}(\mathbf{k}) &= \frac{\hbar^2\mathbf{k}^2}{2M}\Psi_{\mathrm{g}}(\mathbf{k}) \\ &\quad + \int \mathrm{d}^3k' \sum_\sigma \zeta^*_\sigma(\mathbf{k}') a^\dagger_\sigma(\mathbf{k}') \Psi_{\mathrm{e}}(\mathbf{k}+\mathbf{k}') \end{aligned}\tag{3}$$

$$\begin{aligned}\mathrm{i}\hbar\dot{a}_\sigma(\mathbf{k}) &= \hbar\omega_k a_\sigma(\mathbf{k}) \\ &\quad + \zeta_\sigma(\mathbf{k}) \int \mathrm{d}^3k' \Psi^\dagger_{\mathrm{g}}(\mathbf{k}') \Psi_{\mathrm{e}}(\mathbf{k}+\mathbf{k}'). \end{aligned}\tag{4}$$

We have neglected the interatomic interaction terms.

To address the question of self-stabilization consider the following situation: the atoms in the internal ground-state have formed a BEC which is described by a macroscopically occupied coherent collective wavefunction $\Psi_{\mathrm{g}}^{\mathrm{coh}}$. They interact with a traveling wave or standing wave laser which is described by a coherent c-number field $a_\sigma^{\mathrm{coh}}(\mathbf{k})$. Due to this interaction a part of the BEC is coherently excited. We denote the wavefunction for coherently excited atoms by $\Psi_{\mathrm{e}}^{\mathrm{coh}}$. Since both the ground-state BEC and the laser beam are described by c-number fields it is easy to see from equation (2) that $\Psi_{\mathrm{e}}^{\mathrm{coh}}$ must be a c-number field, too. It is only through the spontaneous decay of these coherently excited atoms that q-number deviations from c-number solutions to equations (2–4) can appear. The corresponding SE rate determines the stability of the macroscopic solution.

Let us start with the assumption that the BEC can indeed stabilize itself against SE. In that case a stationary macroscopic solution $(\Psi_{\mathrm{g}}^{\mathrm{coh}}, \Psi_{\mathrm{e}}^{\mathrm{coh}}, a_\sigma^{\mathrm{coh}})$ of equations (2–4) should exist. The problem then can be divided into two separate parts. We first search for the stationary macroscopic coherent solution which includes all interaction effects between atoms and photons beside SE. Having found this solution we can perform a stability analysis to analyze the quantum fluctuations (SE) around it. Spontaneous decay will make the coherent solution unstable and the corresponding quantum corrections will become important on a time scale comparable to the atomic lifetime (which is to be calculated). For times shorter than this lifetime the deviations from the coherent solution will be small (*i.e.*, there are only few non-condensed atoms and non-laser photons).

Given a stationary macroscopic solution of the Heisenberg equations of motion the stability analysis can be performed by applying Bogoliubov's method. This is done by writing the quantum field operators in the form

$$\Psi_{\mathrm{g}}(\mathbf{k}) = \exp[-\mathrm{i}\mu t]\{\Psi_{\mathrm{g}}^{\mathrm{coh}}(\mathbf{k}) + \delta\Psi_{\mathrm{g}}(\mathbf{k})\} \tag{5}$$

$$\Psi_{\mathrm{e}}(\mathbf{k}) = \exp[-\mathrm{i}(\mu+\omega_{\mathrm{L}})t]\{\Psi_{\mathrm{e}}^{\mathrm{coh}}(\mathbf{k}) + \delta\Psi_{\mathrm{e}}(\mathbf{k})\} \tag{6}$$

$$a_\sigma(\mathbf{k}) = \exp[-\mathrm{i}\omega_{\mathrm{L}} t]\{a_\sigma^{\mathrm{coh}}(\mathbf{k}) + \delta a_\sigma(\mathbf{k})\} \tag{7}$$

and retaining in equations (2–4) only terms linear in $\delta\Psi_i$ and δa_σ, which describe the quantum fluctuations around the coherent solution.

The resulting linearized equations of motions are given by

$$\begin{aligned}\mathrm{i}\hbar\delta\dot{\Psi}_{\mathrm{e}}(\mathbf{k}) &= -\hbar\Delta_{\mathrm{L}}\delta\Psi_{\mathrm{e}}(\mathbf{k}) \\ &\quad + \int \mathrm{d}^3k' \Psi_{\mathrm{g}}^{\mathrm{coh}}(\mathbf{k}-\mathbf{k}') \delta a(\mathbf{k}') \zeta(\mathbf{k}') \end{aligned}\tag{8}$$

$$\mathrm{i}\hbar\delta\dot{\Psi}_{\mathrm{g}}(\mathbf{k}) = \int \mathrm{d}^3k' \Psi_{\mathrm{e}}^{\mathrm{coh}}(\mathbf{k}+\mathbf{k}') \delta a^\dagger(\mathbf{k}') \zeta(\mathbf{k}') \tag{9}$$

$$\begin{aligned}\mathrm{i}\hbar\delta\dot{a}(\mathbf{k}) &= \hbar(c|\mathbf{k}| - \omega_{\mathrm{L}})\delta a(\mathbf{k}) \\ &\quad + \zeta(\mathbf{k}) \int \mathrm{d}^3k' \left\{\Psi_{\mathrm{e}}^{\mathrm{coh}}(\mathbf{k}+\mathbf{k}') \delta\Psi_{\mathrm{g}}^\dagger(\mathbf{k}') \right. \\ &\quad \left. + \delta\Psi_{\mathrm{e}}(\mathbf{k}+\mathbf{k}') \Psi_{\mathrm{g}}^{\mathrm{coh}*}(\mathbf{k}')\right\}. \end{aligned}\tag{10}$$

Here ω_{L} is the laser's frequency and $\Delta_{\mathrm{L}} := \omega_{\mathrm{L}} - \omega_{\mathrm{res}}$ its detuning. The linearized equations are valid as long as the photon-atom quantum fluctuations remain small enough, *i.e.*, there are only few non-condensed atoms and non-laser photons. This is certainly the the case for short times. Furthermore several other approximations have been made. First, it is not difficult to see that for a BEC of density 10^{14} cm^{-3} the chemical potential $\hbar\mu$, the kinetic energy $\hbar^2\mathbf{k}^2/(2M)$ of an atom, and the laser's Rabi frequency $\Omega = a^{\mathrm{coh}}\zeta_{\sigma_0}(\mathbf{k}_{\mathrm{L}})/\hbar$ are typically much smaller than the interaction energy $\zeta_\sigma(\mathbf{k})\Psi_{\mathrm{g}}^{\mathrm{coh}}(\mathbf{k})$ if $|\mathbf{k}|$ is of the order of ω_{res}/c. We thus have neglected all terms in which these quantities do appear.

In addition, we have introduced two specific polarization vectors for the electromagnetic field. A "non-coupled" polarization vector $\boldsymbol{\varepsilon}_{\mathrm{NC}}(\mathbf{k})$, which is perpendicular to the photon momentum $\mathbf{k}$ and the atomic dipole moment $\mathbf{d}$, and a coupled polarization vector $\boldsymbol{\varepsilon}_{\mathrm{C}}(\mathbf{k})$, which is perpendicular to $\mathbf{k}$ and $\boldsymbol{\varepsilon}_{\mathrm{NC}}(\mathbf{k})$ (see Fig. 1). Since the interaction is proportional to the scalar product of the polarization vector and $\mathbf{d}$ only electromagnetic modes with polarization $\boldsymbol{\varepsilon}_{\mathrm{C}}(\mathbf{k})$ do interact with the atoms. We associate with these modes the quantum fluctuation operator

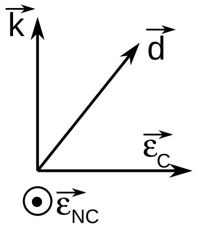

Fig. 1. The polarization vectors can be chosen in a way that only one of them, $\boldsymbol{\varepsilon}_{\rm C}(\mathbf{k})$, is not orthogonal to $\mathbf{d}$. In this case the three vectors $\mathbf{d}$, $\mathbf{k}$, and $\boldsymbol{\varepsilon}_{\rm C}(\mathbf{k})$ are in the same plane. The second polarization vector $\boldsymbol{\varepsilon}_{\rm NC}(\mathbf{k})$ is perpendicular to this plane.

$\delta a(\mathbf{k}) := \delta a_{\sigma={\rm C}}(\mathbf{k})$. It is easy to see that the scalar product $\boldsymbol{\varepsilon}_{\rm C}(\mathbf{k})\cdot\mathbf{d}$, which appears in the definition of $\zeta_\sigma(\mathbf{k})$, is given by $|\mathbf{d}|\sin\vartheta_{\mathbf{k}}$, where $\vartheta_{\mathbf{k}}$ is the angle between $\mathbf{k}$ and $\mathbf{d}$. For notational convenience we have defined $\zeta(\mathbf{k}) := \zeta_{\sigma={\rm C}}(\mathbf{k})$.

It is possible to derive equations (8–10) from an effective Hamiltonian for the quantum fluctuations,

$$H_{\rm fluct} = H_{\rm pol} + H_{\rm spont}. \tag{11}$$

The first part,

$$\begin{aligned} H_{\rm pol} = \hbar \int {\rm d}^3k \left\{ -\Delta_{\rm L}\delta\Psi_{\rm e}^\dagger \delta\Psi_{\rm e} + (c|\mathbf{k}| - \omega_{\rm L})\delta a^\dagger \delta a \right\} \\ + \int {\rm d}^3k{\rm d}^3k' \zeta(\mathbf{k})\Psi_{\rm g}^{\rm coh}(\mathbf{k}-\mathbf{k}') \\ \times \left\{ \delta a(\mathbf{k})\delta\Psi_{\rm e}^\dagger(\mathbf{k}') + \delta a^\dagger(\mathbf{k})\delta\Psi_{\rm e}(\mathbf{k}') \right\}, \end{aligned} \tag{12}$$

conserves the number of photons plus excited atoms,

$$N_{\rm pol} = \int {\rm d}^3k \{\delta\Psi_{\rm e}^\dagger\delta\Psi_{\rm e} + \delta a^\dagger \delta a\}. \tag{13}$$

The first integral in $H_{\rm pol}$ describes the energy of free incoherent photons and atoms. The second integral represents the excitation of atoms from the ground-state BEC and the reabsorption of incoherent photons by the BEC. Its eigenmodes $|\mathbf{q}, r\rangle = \mathcal{P}_{\mathbf{q},r}^\dagger|0\rangle$ are generally superpositions of photons and excited atoms, *i.e.*, polaritons [5]. They are characterized by a continuous, momentum-like quantum number $\mathbf{q}$ and discrete quantum numbers r (see below) and can generally be written as

$$\mathcal{P}_{\mathbf{q},r}^\dagger = \int {\rm d}^3k \{\mathcal{E}_{\mathbf{q},r}(\mathbf{k})\delta\Psi_{\rm e}^\dagger(\mathbf{k}) + \mathcal{A}_{\mathbf{q},r}(\mathbf{k})\delta a^\dagger(\mathbf{k})\} \tag{14}$$

$$\delta\Psi_{\rm e}(\mathbf{k}) = \int {\rm d}^3q \sum_r \mathcal{E}_{\mathbf{q},r}(\mathbf{k})\mathcal{P}_{\mathbf{q},r} \tag{15}$$

$$\delta a(\mathbf{k}) = \int {\rm d}^3q \sum_r \mathcal{A}_{\mathbf{q},r}(\mathbf{k})\mathcal{P}_{\mathbf{q},r}\ . \tag{16}$$

The form of the expansion coefficients $\mathcal{E}_{\mathbf{q},r}(\mathbf{k}), \mathcal{A}_{\mathbf{q},r}(\mathbf{k})$ depends on the particular physical situation and is derived for a traveling and standing-wave laser in Appendices A.2 and B.2, respectively. The second part of the effective Hamiltonian is given by

$$\begin{aligned} H_{\rm spont} = \int {\rm d}^3k{\rm d}^3k' \zeta(\mathbf{k}')\Psi_{\rm e}^{\rm coh}(\mathbf{k}+\mathbf{k}') \\ \times \left\{ \delta a^\dagger(\mathbf{k}')\delta\Psi_{\rm g}^\dagger(\mathbf{k}) + \delta a(\mathbf{k}')\delta\Psi_{\rm g}(\mathbf{k}) \right\}. \end{aligned} \tag{17}$$

It does not conserve $N_{\rm pol}$ and describes the spontaneous decay of coherently excited atoms. If this term would vanish the macroscopic coherent state would be stable against spontaneous decay.

3 General derivation of SE rates

The stability analysis essentially comprises to solve the time evolution of the polariton modes for relatively short times during which the occupation of the macroscopic coherent solution does not change very much. This will allow us to derive the initial SE rate of coherently excited atoms. We assume that initially all atoms and photons are in the state determined by the macroscopic coherent solution, or in other words, the quantized polariton field (photon-atom quantum fluctuations) is initially in the vacuum $|0\rangle$. This state then evolves under the action of the fluctuation Hamiltonian (11) into the time dependent state $|\psi(t)\rangle$.

To describe this time evolution we rewrite the polariton Hamiltonian (12) in the convenient form

$$H_{\rm pol} = \int {\rm d}^3q \sum_r \hbar(\omega_{\mathbf{q},r} - \Delta_{\rm L})\mathcal{P}_{\mathbf{q},r}^\dagger\mathcal{P}_{\mathbf{q},r}, \tag{18}$$

where $\omega_{\mathbf{q},r} - \Delta_{\rm L}$ are the eigenfrequencies of $H_{\rm pol}$. Using equations (15, 16) one also can derive

$$\begin{aligned} H_{\rm spont} = \int {\rm d}^3k \int {\rm d}^3q \sum_r \Big\{ \delta\Psi_{\rm g}(\mathbf{k})\mathcal{P}_{\mathbf{q},r}\, g_{\mathbf{q},r}(\mathbf{k}) \\ + \delta\Psi_{\rm g}^\dagger(\mathbf{k})\mathcal{P}_{\mathbf{q},r}^\dagger\, g_{\mathbf{q},r}^*(\mathbf{k}) \Big\} \end{aligned} \tag{19}$$

with

$$g_{\mathbf{q},r}(\mathbf{k}) := \int {\rm d}^3k' \zeta(\mathbf{k}')\Psi_{\rm e}^{\rm coh}(\mathbf{k}+\mathbf{k}')\mathcal{A}_{\mathbf{q},r}(\mathbf{k}'). \tag{20}$$

To describe the evolution of the state $|\psi(t)\rangle$ we make the following ansatz, which corresponds to the one-photon approximation,

$$\begin{aligned} |\psi(t)\rangle \approx R(t)|0\rangle \\ + \int {\rm d}^3k \int {\rm d}^3q \sum_r S_{\mathbf{q},r}(\mathbf{k},t)\mathcal{P}_{\mathbf{q},r}^\dagger\delta\Psi_{\rm g}^\dagger(\mathbf{k})\ |0\rangle. \end{aligned} \tag{21}$$

The Schrödinger equation ${\rm i}\hbar|\dot\psi\rangle = H_{\rm fluct}|\psi\rangle$ then can be solved by using the Laplace transform $\bar R(s) = \int_0^\infty \exp[-ts]R(t){\rm d}t$ and similarly for $S_{\mathbf{q},s}(\mathbf{k},t)$. The resulting equations,

$${\rm i}\hbar(s\bar R(s) - R(0)) = \int {\rm d}^3k \int {\rm d}^3q \sum_r \bar S_{\mathbf{q},r}(\mathbf{k},s)g_{\mathbf{q},r}(\mathbf{k}) \tag{22}$$

$${\rm i}\hbar s\bar S_{\mathbf{q},r}(\mathbf{k},s) = \hbar(\omega_{\mathbf{q},r} - \Delta_{\rm L})\bar S_{\mathbf{q},r}(\mathbf{k},s) + \bar R(s)g_{\mathbf{q},r}^*(\mathbf{k}), \tag{23}$$

have the solution

$$\bar{R}(s) = \frac{R(0)}{s - I(s)} \cdot \quad (24)$$

The dependence on the particular physical situation is completely determined by the integral

$$I = \frac{1}{\mathrm{i}\hbar^2} \int \mathrm{d}^3 q \sum_r \frac{\int \mathrm{d}^3 k |g_{\mathbf{q},r}(\mathbf{k})|^2}{z_s - \omega_{\mathbf{q},r}} \cdot \quad (25)$$

For notational convenience we have defined the complex variable

$$z_s := \mathrm{i}s + \Delta_{\mathrm{L}}. \quad (26)$$

The most important aspect of I is its complex analytical structure. This is because the inverse Laplace transform is defined by

$$R(t) = \frac{1}{2\pi\mathrm{i}} \int_{\epsilon - \mathrm{i}\infty}^{\epsilon + \mathrm{i}\infty} \mathrm{e}^{ts} \bar{R}(s) \mathrm{d}s, \quad (27)$$

where ϵ is chosen so that the path of integration lies to the right of any branch cuts and poles of $\bar{R}(s)$. From equation (24) it becomes clear that the branch cuts of $\bar{R}(s)$ are those of I and that the poles of $\bar{R}(s)$ essentially depend on the form of I. Assuming that all poles of $\bar{R}(s)$ are simple poles we then find for $R(t)$

$$R(t) = \sum_{s_i} \mathrm{e}^{ts_i} \mathrm{Res}(\bar{R}(s), s_i) + \sum_{\mathcal{B}_j} \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{B}_j} \mathrm{e}^{ts} \bar{R}(s) \mathrm{d}s, \quad (28)$$

where s_i denote the poles of $\bar{R}(s)$ and $\mathcal{B}_j$ the branch cuts. Each pole corresponds to a fraction of the coherently excited atoms which decays (or increases exponentially) with a SE rate of

$$\gamma_i = -2\mathrm{Re}(s_i). \quad (29)$$

The integration contours around the branch cuts corresponds to a fraction of coherently excited atoms with a non-exponential time evolution.

With equation (29) we have found a general expression for the SE rates that can appear in the presence of a BEC. We now want to study the different physical situations of a BEC in a traveling or standing wave laser and to derive the corresponding values of γ_i. To do so we have to find closed expressions for the polariton eigenfrequencies $\omega_{\mathbf{q},r}$ and the functions $g_{\mathbf{q},r}(\mathbf{k})$ in order to derive $I(s)$. These quantities in turn require the knowledge of both the polariton eigenmodes and the fields $(\Psi_{\mathrm{g}}^{\mathrm{coh}}, \Psi_{\mathrm{e}}^{\mathrm{coh}}, a_\sigma^{\mathrm{coh}})$ comprising the macroscopic coherent solution. Since the calculations leading to a closed expression for I are quite involved we present them in the appendices. In the next two sections we analyse the results and give physical interpretations of the effects involved.

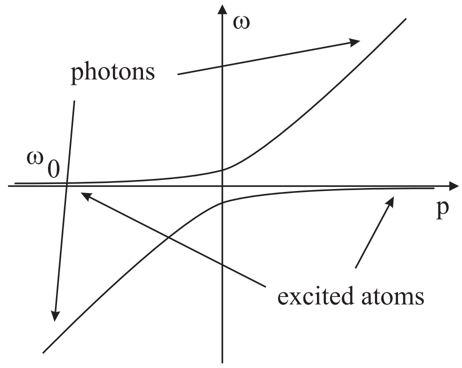

Fig. 2. A homogeneous BEC induces an avoided crossing in the polariton spectrum. Far away from the avoided crossing the polaritons describe excited atoms or photons. Thus, if one focuses on the photons, the avoided crossing provides an effective band gap.

4 Spontaneous emission rates for a BEC in a traveling wave laser

In the case of a BEC interacting with a traveling wave laser, the polariton dispersion relation derived in Appendix A.2 does contain an avoided crossing around the resonance frequency of the atoms (see Fig. 2).

Since nearly resonant photons provide the dominant contribution to SE, it is physically evident that this avoided crossing in the dispersion spectrum will produce an effect on the SE rate which is similar to what a PBG can do.

This effect can be studied by analysing the closed expression for the renormalized value of $I(s)$ which we have derived using the generalized Wigner-Weisskopf approximation presented in Appendix A.3 We find

$$I^{\mathrm{Ren}} = \left(1 - \frac{4\nu_{\mathrm{g}}}{5z_s}\right) I_0^{\mathrm{Ren}} + \frac{N_{\mathrm{e}}\gamma_{\mathrm{vac}}}{5\pi\mathrm{i}} \left\{ -\frac{47\nu_{\mathrm{g}}}{15z_s} + \frac{8}{3} + \frac{z_s}{\nu_{\mathrm{g}}} + \left(1 + \frac{4\nu_{\mathrm{g}}}{z_s}\right)\left(1 - \frac{z_s}{\nu_{\mathrm{g}}}\right)^{3/2} \mathrm{arcoth}\left(\sqrt{1 - \frac{z_s}{\nu_{\mathrm{g}}}}\right) \right\} \cdot \quad (30)$$

In this expression $\gamma_{\mathrm{vac}} := \mathbf{d}^2\omega_{\mathrm{res}}^3/(3\pi\hbar\varepsilon_0 c^3)$ denotes the SE rate in free space and $N_{\mathrm{e}} := V\rho_{\mathrm{e}}$ the number of coherently excited atoms (V is the quantization volume). The frequency

$$\nu_{\mathrm{g}} := \frac{|\mathbf{d}|^2 \rho_{\mathrm{g}}}{2\hbar\varepsilon_0}, \quad (31)$$

with ρ_{g} being the density of atoms in the ground-state, determines the strength of the interaction between photons and excited atoms mediated by the BEC. Typically we have $E_{\mathrm{interaction}} = \hbar\sqrt{\nu_{\mathrm{g}}\omega_{\mathrm{res}}}$ (see Appendix A.2).

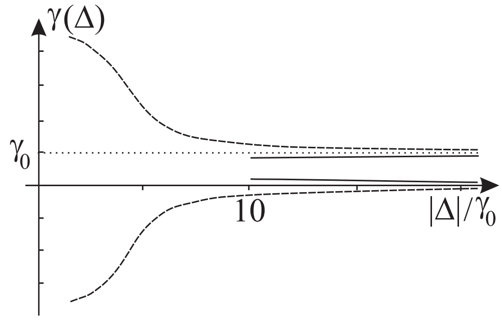

Fig. 3. Spontaneous emission rate of a partially excited BEC in a running laser wave for detuning $\Delta_L > 4\nu_g$ (solid lines) and $\Delta_L < 0$ (dashed lines). The atoms break up into different fractions with different decay rates. The dominating fraction is the one whose decay rate asymptotically approaches γ_{vac}.

As already mentioned in Section 3 the time evolution of the macroscopic coherent solution depends essentially on the analytical structure of $\bar{R}(s)$ of equation (24). In general, $\bar{R}(s)$ has several poles and a branch cut originating from the term including the arcoth in I^{Ren}. This cut lies between $z_s = 0$ and $z_s = \nu_g$. Another important property of I^{Ren} is that it depends on z_s only through the ratio z_s/ν_g so that the magnitude of the SE modification depends on this ratio, too. In addition, a numerical evaluation of equation (30) shows that it is a slowly varying function of the order $\gamma_{vac}N_e$ unless z_s/ν_g is close to zero. This has the following consequences.

In free space z_s can be taken to be close to the pole of $\bar{R}(s)$, *i.e.*, of the order of $\gamma_{vac}N_e$. Hence, the magnitude of non-Markovian effects is essentially determined by the ratio

$$\frac{z_s}{\nu_g} \approx \frac{N_e\gamma_{vac}}{\nu_g} = \frac{16\pi^2}{3}\frac{N_e}{\rho_g\lambda^3}\,. \tag{32}$$

This ratio is proportional to the total number of excited atoms divided by the number of condensed ground-state atoms in a cube whose edges have the length of the optical wavelength λ. The request that this ratio should be small has important consequences when applied to BECs of a density in the order of 10^{14} cm^{-3}. In this case the BEC-induced effects can only be relevant if there are very few coherently excited atoms (in the order of one). However, for higher densities significant effects can occur also for a higher number of excited atoms.

The request that N_e is of order one also implies $|\Delta_L| \gg \Omega_L$ since otherwise the Rabi frequency would be large enough to excite many atoms. As discussed in Appendix A.1, the macroscopic coherent solution implies in this case for $\Delta_L > 0$ the additional constraint $\Delta_L > 4\nu_g$. In Figure 3 the real part of the two dominating poles s_1, s_2 of $I(s)$ is shown as a function of Δ_L for the case $N_e = 1$ and $\nu_g = 2.5\gamma_{vac}$ (corresponding to an atom density ρ_g of 5×10^{14} cm^{-3}). For $\Delta_L > 0$ a third pole appears with a very small negative decay rate ($< 10^{-3}\gamma_{vac}$). The occurrence of negative decay rates is consistent within the range of validity of the linearized equations for the quantum fluctuations and may indicate the formation of an atom-photon bound state [13]. Obviously the change in the SE rate can be quite large for small $|\Delta_L|$. According to equation (28) the fraction of atoms belonging to the poles can be easily calculated by determining the residue at the poles. It turns out that the pole whose real part asymptotically approaches γ_{vac} always dominates and that the fraction of atoms belonging to other poles is significant only for small $|\Delta_L|$. The same holds for the fraction corresponding to the branch cut.

If $|\Delta_L| \gg \gamma_{vac}$ holds the dominant pole s_1 can be calculated by perturbation theory. Its real part (the decay rate) is then given by

$$\frac{1}{2}\gamma(\Delta_L) = N_e\frac{\gamma_{vac}}{2}\left(1 - \frac{4\nu_g}{5\Delta_L}\right) + O(\Delta_L^{-2}). \tag{33}$$

We see that the SE rate is altered by a factor of $1 - 4\nu_g/(5\Delta_L)$. It depends on the sign of Δ_L whether SE is increased or decreased. We remark that the reason why SE depends on the detuning is that the coherently excited atoms are driven by the laser field and thus oscillate at the laser frequency ω_L instead of the resonance frequency ω_{res} (see Eq. (6)).

We shortly summarize the results that we have found for a BEC interacting with a running laser beam. For evolution times smaller than the atomic lifetime and for a weak laser beam which only excites a number $N_e = O(1)$ of excited atoms (for BEC densities of 10^{14} cm^{-3}), the spontaneous emission rate is significantly modified by non-Markovian effects. These effects result from an avoided crossing in the polariton spectrum caused by the extended homogeneous BEC in the internal ground state.

In the next section we will examine the corresponding results for a spatially periodic (lattice) BEC in a standing wave laser beam.

5 Spontaneous emission of a BEC in a standing wave laser

A BEC in an optical lattice formed by a standing wave laser beam becomes spatially periodic and thus provides a kind of periodic dielectric. The polariton spectrum will therefore contain band gaps. We have derived the corresponding dispersion relation in Appendix B.2. As is well known from PBGs a band gap around the resonance frequency will lead to non-Markovian effects in the spontaneous emission of an atom. The examination of these effects will answer the question whether a periodic BEC in an optical lattice can stabilize itself against SE.

To determine the SE rate of a BEC in a 1D optical lattice we have again to evaluate the integral I of equation (25). This task is quite involved and is presented in Appendix B.3. Our final analytical form of the renormalized integral I^{Ren} is given by the somewhat lengthy

$$
\begin{aligned}
I^{\rm Ren} &= \frac{\bar{N}_{\rm e}\gamma_{\rm vac}}{2\pi{\rm i}}\left\{2+\ln\left(\frac{\Lambda}{\omega_{\rm res}}\right)\right\}\\
&+\frac{{\rm i}\gamma_{\rm vac}\bar{N}_{\rm e}}{\pi}\left\{\int_1^\infty v{\rm d}v\ln\left[\frac{(f_0(z_s)-v)(\sqrt{v^2+1}-f_0(z_s))+f_1(z_s)}{(f_0(z_s)-v)(\sqrt{v^2-1}-f_0(z_s))+f_1(z_s)}\right]+\int_0^1 v{\rm d}v\ln\left[\frac{(f_0(z_s)-v)(\sqrt{v^2+1}-f_0(z_s))+f_1(z_s)}{(f_0(z_s)-v)(1-v-f_0(z_s))+f_1(z_s)}\right]\right\}\\
&+\frac{{\rm i}\gamma_{\rm vac}\tilde{N}_{\rm e}}{\pi}\frac{\tilde{\nu}_{\rm g}}{z_s}\int_1^\infty\frac{\sqrt{v}{\rm d}v}{f_0(z_s)-v}\left\{(v^2+1)^{1/4}-(v^2-1)^{1/4}-h(v){\rm arctanh}\left[\frac{(v^2+1)^{1/4}}{h(v)}\right]+h(v){\rm arctanh}\left[\frac{(v^2-1)^{1/4}}{h(v)}\right]\right\}\\
&+\frac{{\rm i}\gamma_{\rm vac}\tilde{N}_{\rm e}}{\pi}\frac{\tilde{\nu}_{\rm g}}{z_s}\int_0^1\frac{\sqrt{v}{\rm d}v}{f_0(z_s)-v}\left\{(v^2+1)^{1/4}-(1-v)^{1/2}-h(v){\rm arctanh}\left[\frac{(v^2+1)^{1/4}}{h(v)}\right]+h(v){\rm arctanh}\left[\frac{\sqrt{1-v}}{h(v)}\right]\right\}\cdot \quad (34)
\end{aligned}
$$

expression

see equation (34) above.

In this result we have introduced a couple of new notations. For notational convenience we have defined

$$f_0(z_s) := \frac{z_s+\omega_{\rm res}}{2ck_{\rm L}} - \frac{\omega_{\rm res}}{2ck_{\rm L}}\frac{\bar{\nu}_{\rm g}}{z_s} \quad (35)$$

$$f_1(z_s) := \frac{\tilde{\nu}_{\rm g}}{z_s}\frac{\omega_{\rm res}}{2ck_{\rm L}} \quad (36)$$

as well as the abbreviation

$$h(v) := \sqrt{f_0(z_s) - f_1(z_s)/(f_0(z_s)-v)}.$$

We also introduced a cut-off $\Lambda \approx m_{\rm e}c^2/\hbar$ to regularise the integral ($m_{\rm e}$ is the electron's mass). Two important physical quantities are given by

$$\bar{N}_{\rm e} := \frac{V}{(2\pi)^3}\sum_m (\Psi^{\rm coh}_{{\rm e},2m+1})^2 \quad (37)$$

$$\tilde{N}_{\rm e} := \frac{V}{(2\pi)^3}\sum_m \Psi^{\rm coh}_{{\rm e},2m-1}\Psi^{\rm coh}_{{\rm e},2m+1} \quad (38)$$

where V denotes the quantization volume and the sum runs over the (real) momentum components $\Psi^{\rm coh}_{{\rm e},m}$ of coherently excited atoms. $\bar{N}_{\rm e}$ is simply the total number of excited atoms in the macroscopic coherent field, and $\tilde{N}_{\rm e}$ describes how these atoms are distributed in momentum space and is always smaller than $\bar{N}_{\rm e}$. It is a measure for the degree of periodicity of the density of excited atoms, very roughly we have $V\rho_{\rm e}(z) \approx \bar{N}_{\rm e} + \tilde{N}_{\rm e}\cos(2zk_{\rm L})$.

The influence of the BEC in a standing wave laser on the SE rate is determined by the frequencies

$$\bar{\nu}_{\rm g} := \frac{\zeta^2(\mathbf{k}_{\rm L})}{\hbar^2\omega_{\rm res}}\bar{\rho}_{\rm g}(2\pi)^3 \approx \frac{\bar{\rho}_{\rm g}\mathbf{d}^2}{2\hbar\varepsilon_0} \quad (39)$$

$$\tilde{\nu}_{\rm g} := \frac{\zeta^2(\mathbf{k}_{\rm L})}{\hbar^2\omega_{\rm res}}\tilde{\rho}_{\rm g}(2\pi)^3 \approx \frac{\tilde{\rho}_{\rm g}\mathbf{d}^2}{2\hbar\varepsilon_0}\cdot \quad (40)$$

In the polariton dispersion relation $\bar{\nu}_{\rm g}$ produces a contribution similar to that of $\nu_{\rm g}$ in the case of a BEC in a traveling wave laser beam (avoided crossing). $\tilde{\nu}_{\rm g}$ produces a real PBG close to the resonance frequency due to the spatial periodicity of a BEC in a 1D optical lattice. The two frequencies define the strength of the interaction mediated by the mean density $\bar{\rho}_{\rm g}$ and the periodic part $\tilde{\rho}_{\rm g}$ of the ground-state BEC,

$$\bar{\rho}_{\rm g} := \frac{1}{(2\pi)^3}\sum_m (\Psi^{\rm coh}_{{\rm g},2m})^2 \quad (41)$$

$$\tilde{\rho}_{\rm g} := \frac{1}{(2\pi)^3}\sum_m \Psi^{\rm coh}_{{\rm g},2m}\Psi^{\rm coh}_{{\rm g},2m+2}. \quad (42)$$

These densities play a similar role to what $\tilde{N}_{\rm e}/V$ and $\bar{N}_{\rm e}/V$ do for coherently excited atoms. For a mean density of $\bar{\rho}_{\rm g} \approx 10^{14}$ cm^{-3} we find $\bar{\nu}_{\rm g} \approx 4\times10^6$ Hz. The magnitude of $\tilde{\nu}_{\rm g}$ can vary between $\bar{\nu}_{\rm g}$ for a very strong optical potential and 0 if the laser beam is switched off.

Although equation (34) has a complicated structure it allows to analyze the main features of $I^{\rm Ren}$ and hence of the time evolution of the macroscopic coherent solution in the presence of (small) quantum fluctuations. It is even possible to estimate the influence of the band gap with some simple arguments.

5.1 General structure of the result

A very important feature of the integral (34) is that all parts of $I^{\rm Ren}$ are proportional to $\bar{N}_{\rm e}\gamma_{\rm vac}$ or $\tilde{N}_{\rm e}\gamma_{\rm vac}$. In addition, it becomes obvious that $I^{\rm Ren}$ depends on the ground-state BEC and on the complex variable z_s essentially through the ratios $z_s/\bar{\nu}_{\rm g}$ and $z_s/\tilde{\nu}_{\rm g}$. The only exception to this is the first term containing z_s in equation (35), but this term is negligible compared to $\omega_{\rm res}$ and does only serve to keep book on which side of the branch cut z_s is placed (see remarks below Eq. (76)).

These facts can be exploited to estimate under which circumstances the influence of the BEC on the SE rate is significant. Since for a BEC in a traveling laser beam the contribution of the poles of $\bar{R}(s)$ usually dominates (see Sect. 4), we will focus on this part. The denominator of $\bar{R}(s)$ is of the form $s - I^{\rm Ren}(s)$, see equation (24). Since a numerical analysis of equation (34) shows that $I^{\rm Ren}(s)$ is of the order of its pre-factors $\bar{N}_{\rm e}\gamma_{\rm vac}$ or $\tilde{N}_{\rm e}\gamma_{\rm vac}$ unless $z_s/\bar{\nu}_{\rm g}$ or $z_s/\tilde{\nu}_{\rm g}$ are small, a pole s_i must be of the order of these pre-factors (if the detuning $\Delta_{\rm L}$ is not very large). In analogy to the case studied in Section 4 one can again infer that the magnitude of the BEC-induced effects essentially depends on ratios of the form $\bar{N}_{\rm e}\gamma_{\rm vac}/\bar{\nu}_{\rm g}$, for instance. As in Section 4 this allows the conclusion that for a BEC with a density in the order of 10^{14} cm^{-3} non-Markovian effects can only be relevant if there are very few coherently excited atoms (in the order of one).

In the case of a BEC in an optical lattice this restriction has additional implications: the Rabi frequency Ω of the coherent standing wave laser has to be very small since otherwise too many atoms would be excited. The number of excited atoms is approximately given by $\bar{N}_{\rm e} \approx (\Omega/\Delta_{\rm L})^2\bar{N}_{\rm g}$, where $\bar{N}_{\rm g}$ denotes the total number of condensed ground-state atoms. Since $\bar{N}_{\rm g}$ is very large the ratio $\Omega/\Delta_{\rm L}$ must be very small in order to achieve $\bar{N}_{\rm e} \approx 1$. This, in turn, means that the optical potential ($\propto \Omega^2/\Delta_{\rm L}$) provided by the standing laser beam is very weak and thus $\tilde{\nu}_{\rm g}$ is much smaller than $\bar{\nu}_{\rm g}$. A small value of $\tilde{\nu}_{\rm g}$ simply means that the polaritonic band gap that is formed in the presence of a BEC will be small and therefore will not have a significant effect on the SE rate.

5.2 Interference channel for spontaneous emission in PBG

Another observation deals with the dependence of $I^{\rm Ren}$ on the wavefunction $\Psi_{\rm e}^{\rm coh}$ of coherently excited atoms. It is known that in free space the shape of the spatial wavefunction of an excited atom does only have a tiny influence on its SE rate [11]. These small corrections are mainly due to the atomic kinetic energy which we have neglected in the Hamiltonian (11) for the quantum fluctuations. In this sense, one would expect that the SE rate in equation (34) does also not depend on the shape of the wavefunction for coherently excited atoms and is proportional to total number $\bar{N}_{\rm e}$ of excited atoms in which this shape does not enter. However, equation (34) does also include the terms proportional to the quantity $\tilde{N}_{\rm e}$ which clearly depend on the shape (for instance, $\tilde{N}_{\rm e}$ vanishes for a spatially homogeneous wavefunction $\Psi_{\rm e}^{\rm coh}$). Principally, this contribution can be as large as that depending on $\bar{N}_{\rm e}$.

This new dependence on the shape of $\Psi_{\rm e}^{\rm coh}$ is an additional effect of the BEC on the SE rate and no consequence of the polaritonic dispersion relation. To understand its origin it is useful to look at equation (67) where $\tilde{N}_{\rm e}$ appears first. This contribution obviously does vanish if $\mathcal{A}_0(\mathbf{q}, r)$ or $\mathcal{A}_{-1}(\mathbf{q}, r)$, which are momentum-components of the photon modes belonging to momentum $\mathbf{q}$ and $\mathbf{q} - 2\mathbf{k}_{\rm L}$, is zero. This is the case for photons interacting with a homogeneous BEC, for instance. The new effect therefore can be considered as arising from the interference between different momentum-components of the photon modes and the wavefunction of coherently excited atoms.

We want to emphasize that this effect is not tied to the presence of a BEC. The only conditions for its existence are the periodicity of both the wavefunction $\Psi_{\rm e}^{\rm coh}$ for excited atoms and the eigenmodes for the photons. Since in an ordinary PBG material the photon eigenmodes are periodic, this new contribution to the SE rate can be present in ordinary PBG materials, too. In an ordinary periodic dielectric the new interference channel for SE even could produce large contributions since the periodicity of the dielectric is produced by, *e.g.*, mechanical forces but not by the light that is used to excite the atoms. Only in the case of a BEC do the standing laser beams play a double role, excitation of atoms and production of a periodicity in the BEC, which results in a suppressed influence of both the polaritonic band gap and the interference channel on the SE rate.

6 Conclusion

We conclude this paper by summarizing the results that we have found. We have examined the self-stabilization of a BEC against SE by performing a stability analysis of a macroscopically occupied state for photons and two-level atoms, which describes a BEC that is coherently coupled to a laser beam. The presence of the ground-state BEC thereby leads to the formation of polaritons and introduces non-Markovian effects in the spontaneous decay of excited atoms.

In the case of a BEC in a traveling-wave laser, the polariton spectrum displays an avoided crossing around the resonance frequency which causes similar changes in the SE rate as a PBG in periodic dielectrics. Its magnitude depends on the ratio $N_{\rm e}/(\rho_{\rm g}\lambda_{\rm L}^3)$ between the total number of excited atoms $N_{\rm e}$ and the number of BEC-atoms inside a cube of the size of an optical wavelength $\lambda_{\rm L}$. If this ratio is much larger than 1 the SE rate will essentially remain unchanged. Otherwise the change can be significant as the numerical examples shown in Figure 3 demonstrate. The change of the SE rate displayed in Figure 3 depends on the detuning of the laser because the coherently excited atoms are driven at the laser's frequency $\omega_{\rm L}$.

For a BEC in a 1D optical lattice two new effects do appear. Being a kind of periodic dielectric the BEC then produces a real polaritonic band gap. The size of this band gap is determined by $\tilde{\nu}_{\rm g}$. As in the case of a traveling wave laser, SE is only significantly altered if there are very few excited atoms. This in turn does imply that the optical lattice must be very weak and therefore produces only a small band gap which has only a very small influence on the SE rate. The second new effect in a periodic BEC is the appearance of a new channel for SE which arises from the interference between different momentum components of the excited-state wavefunction and the photon modes. Though its effect in a BEC is as small as that of the band gap it should also be present in the case of a PBG in an ordinary periodic dielectric where it can be large.

It should be pointed out clearly where exactly the difference between an ordinary periodic dielectric and a BEC in an optical lattice comes into play. In an ordinary dielectric medium the periodicity is produced by whatever forces determine the stability of the medium. The excitation of an atom inside such a medium is done by a light beam, *i.e.*, a completely different physical system. In the case of a BEC in an optical lattice, however, the excitation of the atoms and the potential that produces the periodicity of the BEC both are provided by the same device: the laser beams of the optical lattice. These lattice beams have to achieve two competing goals: to provide a strong periodic potential (to produce a large band gap) and to cause only a weak excitation (to have few excited atoms). As the achievement of both goals is impossible the periodicity of the BEC will only have a tiny influence on the SE rate and it will essentially cause the same effect as a homogeneous BEC in a running laser wave.

This argument also provides the answer to the question whether self-stabilization of a BEC against SE is possible. Since the SE rate is only significantly changed if there are very few excited atoms, and since a large PBG does only form for strong laser beams, a self-stabilization is not possible for BECs with a density in the order of 10^{14} cm^{-3}.

We finally remark that for two reasons our results are not applicable to BECs confined in a micrometer-sized trap, a case discussed in the literature [6,7]. The first reason is that we did not include a trapping potential in our calculations. While principally a potential could be included it would lead to mathematical complications which would make an analytical treatment impossible. Secondly, our work is concerned with BECs which are extended enough to allow the formation of polaritons. This is not the case for current BECs in a relatively small trap. The necessary extension of the BEC can be estimated by considering the typical interaction energy for the formation of polaritons which is given by $\hbar\sqrt{\nu_{\rm g}\omega_{\rm res}}$ (see Appendix A.2). For a BEC with a density of 10^{14} cm^{-3} this energy is in the order of $\hbar \times 10^{11}$ Hz. For the formation of polaritons a photon must therefore be inside the BEC longer than 10^{-11} s. Since it travels at the speed of light the BEC must therefore be larger than about 3 mm.

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Appendix A: BEC in a running laser wave

A.1 Derivation of the macroscopic coherent solution

We are interested in finding a particular solution $(\Psi_{\rm g}^{\rm coh}, \Psi_{\rm e}^{\rm coh}, a_\sigma^{\rm coh}(\mathbf{k}))$ of macroscopically occupied fields to equations (2–4) which describes a BEC coherently coupled to a running laser wave. We thus make the ansatz

$$\Psi_{\rm g}^{\rm coh}(\mathbf{k}) = (2\pi)^{3/2}\sqrt{\rho_{\rm g}}\delta(\mathbf{k})\exp[-{\rm i}\mu t] \quad (43)$$

$$a_\sigma^{\rm coh}(\mathbf{k}) = \exp[-{\rm i}\omega_{\rm L}t]\delta(\mathbf{k}-\mathbf{k}_{\rm L}) \times \delta_{\sigma,\sigma_{\rm L}}\Omega_{\rm L}[2(2\pi)^3\hbar\varepsilon_0\omega_{k_{\rm L}}]^{1/2}/(|\mathbf{d}|\omega_{\rm res}) \quad (44)$$

$$\Psi_{\rm e}^{\rm coh}(\mathbf{k}) = (2\pi)^{3/2}\sqrt{\rho_{\rm e}}\delta(\mathbf{k}-\mathbf{k}_{\rm L})\exp[-{\rm i}(\mu+\omega_{\rm L})t] \quad (45)$$

which corresponds to a homogeneous ground-state BEC of density $\rho_{\rm g}$, a laser beam with frequency $\omega_{\rm L}$, Rabi frequency $\Omega_{\rm L} > 0$, polarization $\sigma_{\rm L}$, and wave-vector $\mathbf{k}_{\rm L} = k_{\rm L}\mathbf{e}_z$ (*inside* the BEC), and coherently excited atoms of density $\rho_{\rm e}$ and of momentum $\hbar\mathbf{k}_{\rm L}$. Inserting these expressions into the Heisenberg equations of motions leads to a set of algebraical conditions which fix the chemical potential $\hbar\mu$, the laser wavenumber $k_{\rm L}$, and the density of coherently excited atoms $\rho_{\rm e}$ which we assume to be smaller than $\rho_{\rm g}$. If we neglect the kinetic energy the density of excited atoms is given by $\sqrt{\rho_{\rm e}} = \sqrt{\rho_{\rm g}}\Omega_{\rm L}/(\mu+\Delta_{\rm L})$. The wavenumber $k_{\rm L}$ is fixed by $ck_{\rm L} = \omega_{\rm L}/2 + \sqrt{(\omega_{\rm L}/2)^2 - \omega_{\rm res}^2\nu_{\rm g}/(\mu+\Delta_{\rm L})}$, where $\nu_{\rm g}$ is defined in equation (31). Note that $k_{\rm L}$ generally is different from the free-space value $\omega_{\rm L}/c$. For $\Delta_{\rm L} \le 0$ ($\Delta_{\rm L} \ge 0$) the chemical potential is given by $\mu = -\Delta_{\rm L}/2 \pm \sqrt{(\Delta_{\rm L}/2)^2 + \Omega_{\rm L}^2}$ which implies $\mu + \Delta_{\rm L} > 0$ ($\mu + \Delta_{\rm L} < 0$), respectively. Note that for $\Delta_{\rm L} > 0$ the expression for $ck_{\rm L}$ implies the additional constraint $\mu + \Delta_{\rm L} > 4\nu_{\rm g}$.

A.2 Derivation of polariton eigenmodes

Having found the macroscopic coherent solution we are in the position to derive the polariton eigenmodes. Because of the delta distribution appearing in the macroscopic solution (43) the Hamiltonian (12) reduces to a sum of two-level systems so that its eigenmodes are quite easy to find. They consist of polaritons with momentum $\hbar\mathbf{q}$ and frequency spectrum

$$\omega_{\mathbf{q},\pm} = \frac{\Delta_q}{2} \pm W_q, \quad (46)$$

where we have defined $W_q := \sqrt{(\Delta_q/2)^2 + \nu_{\rm g}\omega_{\rm res}\sin^2\vartheta_{\mathbf{q}}}$ and $\Delta_q := c|\mathbf{q}| - \omega_{\rm res}$. The coefficients of the polariton creation operator in equation (14) are given by

$$\mathcal{E}_{\mathbf{q},\pm}(\mathbf{k}) = \frac{\delta(\mathbf{q}-\mathbf{k})}{\sqrt{2W_q}}\frac{\sqrt{\nu_{\rm g}\omega_{\rm res}}\sin\vartheta_{\mathbf{q}}}{\sqrt{W_q \pm \Delta_q/2}},$$

$$\mathcal{A}_{\mathbf{q},\pm}(\mathbf{k}) = \pm\frac{\delta(\mathbf{q}-\mathbf{k})}{\sqrt{2W_q}}\sqrt{W_q \pm \Delta_q/2} \quad (47)$$

so that we find for equation (20)

$$g_{\mathbf{q},r}(\mathbf{k}) = (2\pi)^{3/2}\zeta(\mathbf{q}) \times \sqrt{\rho_{\rm e}}\delta(\mathbf{k}+\mathbf{q}-\mathbf{k}_{\rm L})\frac{\omega_{\mathbf{q},r}}{\sqrt{\omega_{\mathbf{q},r}^2 + \nu_{\rm g}\omega_{\rm res}\sin^2\vartheta_{\mathbf{q}}}}\cdot \quad (48)$$

The polariton spectrum $\omega_{\pm,\mathbf{q}}$ clearly exhibits an avoided crossing around $\Delta_q = 0$ of width $\sqrt{\nu_{\rm g}\omega_{\rm res}}\sin\vartheta_{\mathbf{q}}$ (see Fig. 2). It also contains a small gap whose edge is reached in the limit $|\mathbf{q}| \to 0$ and ∞ [5]. This gap will play no role for the SE rate since its edges are far away from the resonance frequency. This is physically reasonable since far away from resonance the two-level approximation for the atomic internal structure ceases to be valid.

A.3 Calculation of I(s)

With the help of equation (48) the integral $I(s)$ of equation (25) can be written in the form

$$I = \frac{V\nu_e\omega_{\rm res}^2}{(2\pi)^3}\int d^3k \frac{\sin^2\vartheta_{\mathbf{k}}}{\omega_{\mathbf{k}}\left(s - i\omega_L + i\omega_{\mathbf{k}} + \frac{\nu_g\omega_{\rm res}\sin^2\vartheta_{\mathbf{k}}}{s-i\Delta_L}\right)}, \tag{49}$$

where V denotes the quantization volume. This integral agrees with the one found in absence of a BEC (which describes SE in free space) by setting $\nu_g = \Delta_L = 0$. We denote this free space integral by $I_0 := I(\nu_g = \Delta_L = 0)$. Both integrals are linearly divergent and can be treated in the way pointed out by Bethe (see, *e.g.* Ref. [12]), *i.e.*, we renormalise the integrals by subtracting the free-electron contribution,

$$I^{\rm Ren} := I - \frac{V\nu_e\omega_{\rm res}^2}{(2\pi)^3 i}\int d^3k \frac{\sin^2\vartheta_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \tag{50}$$

and $I_0^{\rm Ren} = I^{\rm Ren}(\nu_g = \Delta_L = 0)$. These renormalized integrals are only logarithmically divergent.

At this point it is customary in the calculation of the free-space SE to perform the Wigner-Weisskopf approximation by neglecting the dependence of $I_0^{\rm Ren}(s)$ on s. In the presence of a band gap this is inappropriate due to the strong variation of the mode density around the gap [2,3]. Nevertheless, we can perform generalized Wigner-Weisskopf approximation in the following way. We expect that the typical timescale on which SE happens is much larger than the optical cycle timescale $1/\omega_{\rm res}$. From the definition of the inverse Laplace transform (27) it is clear that the variable s plays more or less the role of a Fourier-transformed time. We thus expect that only values of s much smaller than $\omega_{\rm res}$ contribute significantly to the SE. This implies that we can neglect (the imaginary part of) s wherever it appears together with $\omega_{\rm res}$ or ω_L. Thus, we are allowed to set $s - i\omega_L \approx -i\omega_L$ in the denominator of $I(s)$ while retaining the term depending on $s - i\Delta_L$.

In the case of $I_0^{\rm Ren}$ this procedure immediately reproduces the Wigner-Weisskopf result $I_0^{\rm Ren} \approx N_e\{(\gamma_{\rm vac}/2) + i\Delta_{\rm Lamb}^{\text{2-lev}}\}$, where $\gamma_{\rm vac}$ and N_e are defined in Section 4. To fix $\Delta_{\rm Lamb}^{\text{2-lev}}$ we follow the theory of Bethe (see, *e.g.* [12]) and introduce a cut-off frequency of $m_e c^2/\hbar$ in $I_0^{\rm Ren}$, where m_e is the electron's mass. Calculating the principal value of the integral then leads to $\Delta_{\rm Lamb}^{\text{2-lev}} \approx 2\gamma_{\rm vac}$. In contrast to free space the SE rate in a BEC depends on $\Delta_{\rm Lamb}^{\text{2-lev}}$ since such a radiative frequency correction shifts the center of the avoided crossing (or of a band gap [3]).

It remains to calculate a renormalized expression of the integral $I^{\rm Ren}$ in the presence of a BEC. Fortunately, this task reduces to integrals proportional to $I_0^{\rm Ren}$ and a couple of convergent integrals and leads to equation (30).

Appendix B: BEC in a standing wave laser

B.1 Derivation of the macroscopic coherent solution

Since in a running laser wave the BEC density is not periodic no PBGs can be formed. It is therefore of interest to study a BEC interacting with a standing laser wave so that the formation of photonic, or rather polaritonic band gaps [10], is possible. The coherent laser field describing a standing wave is given by

$$a_\sigma^{\rm coh}(\mathbf{k}) = a_1^{\rm coh}\delta_{\sigma,\sigma_0}\{\delta(\mathbf{k}-\mathbf{k}_L) + \delta(\mathbf{k}+\mathbf{k}_L)\}\exp[-i\omega_L t]. \tag{51}$$

We assume that the amplitude $a_1^{\rm coh}$ is real and that the polarization σ_0 of the laser beam is parallel to the dipole moment $\mathbf{d}$ of the atoms.

Since the laser field provides a periodic potential for the atoms it is reasonable to assume that the macroscopic atomic fields are periodic, too (at least for the ground-state of the system). One also can make the ansatz that $\Psi_g^{\rm coh}$ has period $2k_L$ so that the coherent solutions can be written as

$$\Psi_g^{\rm coh}(\mathbf{k}) = e^{-i\mu t}\sum_n \delta(\mathbf{k}-2n\mathbf{k}_L)\Psi_{g,2n}^{\rm coh} \tag{52}$$

$$\Psi_e^{\rm coh}(\mathbf{k}) = e^{-i(\mu+\omega_L)t}\sum_n \delta(\mathbf{k}-(2n+1)\mathbf{k}_L)\Psi_{e,2n+1}^{\rm coh}. \tag{53}$$

Inserting this into equations (2–4) leads to the matrix equations

$$(\omega_L - ck_L)\Omega = \frac{\zeta_{\sigma_0}^2(\mathbf{k}_L)}{\hbar^2}\sum_n \Psi_{g,2n}^{\rm coh}\Psi_{e,2n+1}^{\rm coh} \tag{54}$$

$$\left(\Delta_L + \mu - (2n+1)^2\frac{\hbar\mathbf{k}_L^2}{2M}\right)\Psi_{e,n+1}^{\rm coh} = \Omega\{\Psi_{g,2n}^{\rm coh} + \Psi_{g,2n+2}^{\rm coh}\} \tag{55}$$

$$\left(\mu - (2n)^2\frac{\hbar\mathbf{k}_L^2}{2M}\right)\Psi_{g,2n}^{\rm coh} = \Omega\{\Psi_{e,2n-1}^{\rm coh} + \Psi_{e,2n+1}^{\rm coh}\}. \tag{56}$$

We have assumed that $\zeta_\sigma(\mathbf{k})$ is real and does not depend on the sign of $\mathbf{k}$ and introduced the real Rabi frequency $\Omega := a_1^{\rm coh}\zeta_{\sigma_0}(\mathbf{k}_L)/\hbar$. For consistency with the assumption that $a_1^{\rm coh}$ is a real quantity the coefficients $\Psi_{g,n}^{\rm coh}$ and $\Psi_{e,n}^{\rm coh}$ must be real, too.

The system (54–56) of algebraic equations can easily be solved numerically. To do so we assume that the Rabi frequency of the laser beam is a given quantity. For a given value of k_L the two equations (55, 56) then just describe the well-known problem of a two-level atom moving in a standing laser wave. This is a simple system of linear equations and can be solved in a standard manner. The solution then can be inserted into equation (54) which then, because ω_L and $a_1^{\rm coh}$ are fixed, determines the value of k_L. We then have reinserted the new value for k_L into the system (54–56) and iterated the procedure until k_L did not change significantly anymore.

$$\mathcal{A}_0(\mathbf{q},r) = \frac{\omega_{\mathbf{q},r}\sqrt{\omega_{\mathbf{q},r}^2 - \omega_{\mathbf{q},r}\Delta_{-1} - \bar{\nu}_{\rm g}\omega_{\rm res}}}{\sqrt{2\tilde{\nu}_{\rm g}^2\omega_{\rm res}^2 + (\omega_{\mathbf{q},r}^2 + \bar{\nu}_{\rm g}\omega_{\rm res})(2\omega_{\mathbf{q},r}^2 - \omega_{\mathbf{q},r}(\Delta_0 + \Delta_{-1}) - 2\bar{\nu}_{\rm g}\omega_{\rm res})}} \tag{64}$$

$$\mathcal{A}_{-1}(\mathbf{q},r) = \frac{\omega_{\mathbf{q},r}\tilde{\nu}_{\rm g}\omega_{\rm res}/\sqrt{\omega_{\mathbf{q},r}^2 - \omega_{\mathbf{q},r}\Delta_{-1} - \bar{\nu}_{\rm g}\omega_{\rm res}}}{\sqrt{2\tilde{\nu}_{\rm g}^2\omega_{\rm res}^2 + (\omega_{\mathbf{q},r}^2 + \bar{\nu}_{\rm g}\omega_{\rm res})(2\omega_{\mathbf{q},r}^2 - \omega_{\mathbf{q},r}(\Delta_0 + \Delta_{-1}) - 2\bar{\nu}_{\rm g}\omega_{\rm res})}}. \tag{65}$$

B.2 Derivation of polariton eigenmodes

In this case the periodic structure of the macroscopic solution (52) leads to a more complicated structure of the eigenmodes of $H_{\rm pol}$ than in the case of a traveling laser wave. To find these modes we make in equation (14) the ansatz $\mathcal{E}_{\mathbf{q},r}(\mathbf{k}) = \sum_{m\in\mathbf{Z}} \mathcal{E}_m(\mathbf{q},r)\delta(\mathbf{k} - \mathbf{q} - 2m\mathbf{k}_{\rm L})$ and correspondingly for $\mathcal{A}_{\mathbf{q},r}(\mathbf{k})$. Now $\mathbf{q}$ denotes the quasi-momentum of the polariton. For a single standing laser wave along the z-axis q_z is confined to $[-k_{\rm L}, k_{\rm L}]$ whereas q_x, q_y represent the real momentum of the polariton perpendicular to the laser beam. The index r is a collective notation for discrete quantum numbers which include an internal quantum number taking two values (since two quantum fields δa and $\delta\Psi_{\rm e}$ are involved) and the band index. This ansatz leads to

$$g_{\mathbf{q},r}(\mathbf{k}) := \sum_m \mathcal{A}_m(\mathbf{q},r)\zeta(\mathbf{q}+2m\mathbf{k}_{\rm L})\Psi_{\rm e}^{\rm coh}(\mathbf{k}+\mathbf{q}+2m\mathbf{k}_{\rm L}). \tag{57}$$

and results in the eigenvalue equations

$$E(\mathbf{q},r)\mathcal{E}_m(\mathbf{q},r) = -\hbar\Delta_{\rm L}\mathcal{E}_m(\mathbf{q},r) + \sum_n \Psi_{{\rm g},2n}^{\rm coh}\zeta(\mathbf{q}+2(m-n)\mathbf{k}_{\rm L})\mathcal{A}_{m-n}(\mathbf{q},r) \tag{58}$$

$$E(\mathbf{q},r)\mathcal{A}_m(\mathbf{q},r) = \hbar(c|\mathbf{q}+2m\mathbf{k}_{\rm L}| - \omega_{\rm L})\mathcal{A}_m(\mathbf{q},r) + \sum_n \Psi_{{\rm g},2n}^{\rm coh}\zeta(\mathbf{q}+2m\mathbf{k}_{\rm L})\mathcal{E}_{m+n}(\mathbf{q},r). \tag{59}$$

These equations can be substantially simplified by noting that the frequency difference $\hbar(c|\mathbf{q}+2m\mathbf{k}_{\rm L}| - \omega_{\rm L})$ is huge compared to all other energy scales involved unless $m = 0, \pm 1$ and $\mathbf{q}$ is close to $\pm\mathbf{k}_{\rm L}$. We thus can approximate the photon-part of all modes with $|m| > 1$ as free photons and need only to retain the coefficients $\mathcal{A}_0(\mathbf{q},r)$ and $\mathcal{A}_{-1}(\mathbf{q},r)$ for $q_z \in [0, k_{\rm L}]$ and the coefficients $\mathcal{A}_0(\mathbf{q},r)$ and $\mathcal{A}_1(\mathbf{q},r)$ for $q_z \in [-k_{\rm L}, 0]$, respectively. We will focus here on the case $q_z \in [0, k_{\rm L}]$ since the second case can be treated analogously.

To solve the resulting equations we introduce the two quantities $F_0 := \sum_n \Psi_{{\rm g},2n}^{\rm coh}\mathcal{E}_n(\mathbf{q},r)$ and $F_{-1} := \sum_n \Psi_{{\rm g},2n}^{\rm coh}\mathcal{E}_{n-1}(\mathbf{q},r)$ and make the approximation $\zeta(\mathbf{q}) \approx \zeta(\mathbf{q}-2\mathbf{k}_{\rm L}) \approx \zeta(\mathbf{k}_{\rm L})$ so that the problem is reduced to the simple matrix eigenvalue equation

$$(E(\mathbf{q},r)+\hbar\Delta_{\rm L})\begin{pmatrix} F_0 \\ F_{-1} \\ \mathcal{A}_0 \\ \mathcal{A}_{-1}\end{pmatrix} = \begin{pmatrix} 0 & 0 & \zeta(\mathbf{k}_{\rm L})(2\pi)^3\bar{\rho}_{\rm g} & \zeta(\mathbf{k}_{\rm L})(2\pi)^3\tilde{\rho}_{\rm g} \\ 0 & 0 & \zeta(\mathbf{k}_{\rm L})(2\pi)^3\tilde{\rho}_{\rm g} & \zeta(\mathbf{k}_{\rm L})(2\pi)^3\bar{\rho}_{\rm g} \\ \zeta(\mathbf{k}_{\rm L}) & 0 & \hbar\Delta_0 & 0 \\ 0 & \zeta(\mathbf{k}_{\rm L}) & 0 & \hbar\Delta_{-1}\end{pmatrix}\begin{pmatrix} F_0 \\ F_{-1} \\ \mathcal{A}_0 \\ \mathcal{A}_{-1}\end{pmatrix}. \tag{60}$$

Here we have introduced

$$\Delta_0 := c|\mathbf{q}| - \omega_{\rm res} \tag{61}$$

$$\Delta_{-1} := c|\mathbf{q}-2\mathbf{k}_{\rm L}| - \omega_{\rm res} \tag{62}$$

and the quantities $\bar{\rho}_{\rm g}$ and $\tilde{\rho}_{\rm g}$ which are defined in equations (41, 42), respectively.

The problem of finding the eigenvalues and eigenvectors of a 4×4 matrix is a basic one. The eigenvalues $\omega_{\mathbf{q},r} := (E(\mathbf{q},r)+\hbar\Delta_{\rm L})/\hbar$ fulfill the relation $P_{\rm ch}(\omega_{\mathbf{q},r}) = 0$, where

$$P_{\rm ch}(z) = (z^2 - z\Delta_{-1} - \bar{\nu}_{\rm g}\omega_{\rm res}) \times (z^2 - z\Delta_0 - \bar{\nu}_{\rm g}\omega_{\rm res}) - \tilde{\nu}_{\rm g}^2\omega_{\rm res}^2 \tag{63}$$

is the characteristic polynomial of the matrix and the frequencies $\bar{\nu}_{\rm g}$ and $\tilde{\nu}_{\rm g}$ are defined in equations (39, 40).

Due to the periodicity of the ground-state BEC $\omega_{\mathbf{q},r}$ as a function of the quasi-momentum $\mathbf{q}$ exhibits the phenomenon of band gaps [10], see Figure 4. Though closed expressions for the eigenvalues $\omega_{\mathbf{q},r}$ do exist they are rather cumbersome and not of much use for our problem.

We instead will use a theorem on the eigenvalues to derive the physical quantities of interest. To do so we assume that we already know the eigenvalues. For a given eigenvalue $\omega_{\mathbf{q},r}$ it is easy to solve for the eigenvectors. For the relevant components we find

see equations (64, 65) above.

The normalization has been done by requiring the particle number (13) to be pseudo-normalized, *i.e.*,

$$\sum_m \left\{|\mathcal{E}_m(\mathbf{q},r)|^2 + |\mathcal{A}_m(\mathbf{q},r)|^2\right\} = 1. \tag{66}$$

The analogous equation for $q_z \in [-k_{\rm L}, 0]$ is obtained if $\mathcal{A}_{-1}(\mathbf{q},r)$ is replaced by $\mathcal{A}_1(\mathbf{q},r)$ and Δ_{-1} by $\Delta_1 := c|\mathbf{q}+2\mathbf{k}_{\rm L}| - \omega_{\rm res}$.

$$I = \frac{\bar{N}_{\rm e}}{{\rm i}\hbar^2}\int_{|q_z|>2k_{\rm L}} {\rm d}q_z \int_{-\infty}^{\infty} {\rm d}q_x {\rm d}q_y \frac{\zeta(\mathbf{q})^2}{z_s - c|\mathbf{q}| + \omega_{\rm res}}$$
$$+ \frac{z_s}{{\rm i}\hbar^2}\int_{V_-} \frac{\bar{N}_{\rm e}\zeta(\mathbf{q})^2[z_s^2 - z_s\Delta_1 - \bar{\nu}_{\rm g}\omega_{\rm res}] + \bar{N}_{\rm e}\zeta(\mathbf{q}+2\mathbf{k}_{\rm L})^2[z_s^2 - z_s\Delta_0 - \bar{\nu}_{\rm g}\omega_{\rm res}] + 2\tilde{N}_{\rm e}\tilde{\nu}_{\rm g}\omega_{\rm res}\zeta(\mathbf{q})\zeta(\mathbf{q}+2\mathbf{k}_{\rm L})}{z_s^4 - z_s^3(\Delta_0+\Delta_1) + z_s^2(\Delta_0\Delta_1 - 2\bar{\nu}_{\rm g}\omega_{\rm res}) + z_s\bar{\nu}_{\rm g}\omega_{\rm res}(\Delta_0+\Delta_1) + \bar{\nu}_{\rm g}^2\omega_{\rm res}^2 - \tilde{\nu}_{\rm g}^2\omega_{\rm res}^2}$$
$$+ \frac{z_s}{{\rm i}\hbar^2}\int_{V_+} {\rm d}^3q \frac{\bar{N}_{\rm e}\zeta(\mathbf{q})^2[z_s^2 - z_s\Delta_{-1} - \bar{\nu}_{\rm g}\omega_{\rm res}] + \bar{N}_{\rm e}\zeta(\mathbf{q}-2\mathbf{k}_{\rm L})^2[z_s^2 - z_s\Delta_0 - \bar{\nu}_{\rm g}\omega_{\rm res}] + 2\tilde{N}_{\rm e}\tilde{\nu}_{\rm g}\omega_{\rm res}\zeta(\mathbf{q})\zeta(\mathbf{q}-2\mathbf{k}_{\rm L})}{z_s^4 - z_s^3(\Delta_0+\Delta_{-1}) + z_s^2(\Delta_0\Delta_{-1} - 2\bar{\nu}_{\rm g}\omega_{\rm res}) + z_s\bar{\nu}_{\rm g}\omega_{\rm res}(\Delta_0+\Delta_{-1}) + \bar{\nu}_{\rm g}^2\omega_{\rm res}^2 - \tilde{\nu}_{\rm g}^2\omega_{\rm res}^2} \quad (73)$$

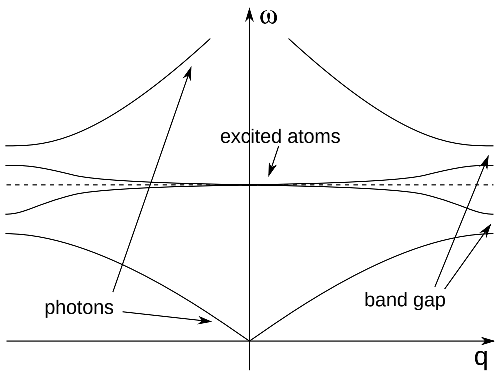

Fig. 4. A schematic drawing of the polariton spectrum for a BEC in a 1D optical lattice. Because the BEC is periodic band gaps do appear.

B.3 Calculation of I(s)

To calculate the integral $I(s)$ of equation (25) we first need to calculate the integral over $\mathbf{k}$ in the nominator which is an easy task. With equation (57) we find for $q_z \in [0, k_{\rm L}]$

$$\int {\rm d}^3k|g_{\mathbf{q},r}(\mathbf{k})|^2 = \bar{N}_{\rm e}\{\mathcal{A}_0^2(\mathbf{q},r)\zeta^2(\mathbf{q}) + \mathcal{A}_{-1}^2(\mathbf{q},r)\zeta^2(\mathbf{q}-2\mathbf{k}_{\rm L})\} + 2\tilde{N}_{\rm e}\mathcal{A}_{-1}(\mathbf{q},r)\mathcal{A}_0(\mathbf{q},r)\zeta(\mathbf{q})\zeta(\mathbf{q}-2\mathbf{k}_{\rm L}) \quad (67)$$

if r is in the lowest two energy bands. The numbers $\bar{N}_{\rm e}$ and $\tilde{N}_{\rm e}$ are defined in equations (37, 38), respectively. For $q_z \in [-k_{\rm L}, 0]$ and the two lowest energy bands one has to replace $\mathcal{A}_{-1}(\mathbf{q},r)$ by $\mathcal{A}_1(\mathbf{q},r)$ and $\zeta(\mathbf{q}-2\mathbf{k}_{\rm L})$ by $\zeta(\mathbf{q}+2\mathbf{k}_{\rm L})$. For all higher bands, which according to our approximation just describe free photons, the integral (67) has the simple value $\bar{N}_{\rm e}\zeta^2(\mathbf{q})$.

From equation (67) it becomes clear that to calculate I of equation (25) we have to find closed expressions for terms like $\sum_r \mathcal{A}_{-1}(\mathbf{q},r)\mathcal{A}_0(\mathbf{q},r)/(z_s - \omega_{\mathbf{q},r})$. For the lowest two energy bands, the sum over r now runs over the four eigenvectors of the matrix (60). It would be extremely tedious if not practically impossible to derive these sums by simply inserting the complicated closed expressions for $\omega_{\mathbf{q},r}$ into them. Instead, we start with the observation that the polynomial appearing in the denominator of equations (64, 65) can be written as

$$2\tilde{\nu}_{\rm g}^2\omega_{\rm res}^2 + (\omega_{\mathbf{q},r}^2 + \bar{\nu}_{\rm g}\omega_{\rm res}) \times (2\omega_{\mathbf{q},r}^2 - \omega_{\mathbf{q},r}(\Delta_0+\Delta_{-1}) - 2\bar{\nu}_{\rm g}\omega_{\rm res}) = \omega_{\mathbf{q},r}P'_{\rm ch}(\omega_{\mathbf{q},r}), \quad (68)$$

where $P'_{\rm ch}(z)$ denotes the derivative of the characteristic polynomial (63). This enables us to write the sum under consideration in the form

$$\sum_r \frac{\mathcal{A}_{-1}(\mathbf{q},r)\mathcal{A}_0(\mathbf{q},r)}{(z_s - \omega_{\mathbf{q},r})} = \tilde{\nu}_{\rm g}\omega_{\rm res}\sum_r \frac{\omega_{\mathbf{q},r}}{P'_{\rm ch}(\omega_{\mathbf{q},r})(z_s - \omega_{\mathbf{q},r})} \,. \quad (69)$$

This can be further simplified by noting that the characteristic polynomial can also be written in the form $P_{\rm ch}(z) = \prod_r (z - \omega_{\mathbf{q},r})$ so that we have $P'_{\rm ch}(\omega_{\mathbf{q},r}) = \prod_{r'\neq r}(\omega_{\mathbf{q},r} - \omega_{\mathbf{q},r'})$. Using this expression it is straightforward if still tedious to find

$$\sum_r \frac{\mathcal{A}_{-1}(\mathbf{q},r)\mathcal{A}_0(\mathbf{q},r)}{(z_s - \omega_{\mathbf{q},r})} = \frac{z_s\tilde{\nu}_{\rm g}\omega_{\rm res}}{P_{\rm ch}(z_s)} \,. \quad (70)$$

We thus have been able to calculate this sum without explicit knowledge of the eigenvalues of equation (60). The remaining sums which result from the insertion of equation (67) into equation (25) can be treated in a similar way and are given by

$$\sum_r \frac{\mathcal{A}_0^2(\mathbf{q},r)}{(z_s - \omega_{\mathbf{q},r})} = z_s\frac{z_s^2 - z_s\Delta_{-1} - \bar{\nu}_{\rm g}\omega_{\rm res}}{P_{\rm ch}(z_s)} \quad (71)$$

$$\sum_r \frac{\mathcal{A}_{-1}^2(\mathbf{q},r)}{(z_s - \omega_{\mathbf{q},r})} = z_s\frac{z_s^2 - z_s\Delta_0 - \bar{\nu}_{\rm g}\omega_{\rm res}}{P_{\rm ch}(z_s)} \quad (72)$$

for the lowest two energy bands and $q_z \in [0, k_{\rm L}]$. Again the corresponding expressions for $q_z \in [-k_{\rm L}, 0]$ are obtained by replacing $\mathcal{A}_{-1}(\mathbf{q},s)$ by $\mathcal{A}_1(\mathbf{q},s)$ and Δ_{-1} by Δ_1.

Taking everything together the use of equation (67) and the three sums (70, 71, 72) allows us to bring the integral (25) into the form

see equation (73) above

where the areas of integration $V_\pm$ are given by $q_x, q_y \in (-\infty, \infty)$ and $q_z \in [0, k_{\rm L}]$ for V_+ as well as $q_z \in [-k_{\rm L}, 0]$ for

$$I - I_0 = \frac{2\mathrm{i}\bar{N}_\mathrm{e}}{\hbar^2}\int_0^{2k_\mathrm{L}} \mathrm{d}q_z \int_{-\infty}^{\infty} \mathrm{d}q_x \mathrm{d}q_y \frac{\zeta(\mathbf{q})^2}{z_s - c|\mathbf{q}| + \omega_\mathrm{res}}$$
$$+ \frac{2z_s}{\mathrm{i}\hbar^2}\int_0^{2k_\mathrm{L}} \mathrm{d}q_z \int \mathrm{d}q_x \mathrm{d}q_y \frac{\bar{N}_\mathrm{e}\zeta(\mathbf{q})^2[z_s^2 - z_s\Delta_{-1} - \bar{\nu}_\mathrm{g}\omega_\mathrm{res}] + 2\tilde{N}_\mathrm{e}\tilde{\nu}_\mathrm{g}\omega_\mathrm{res}\zeta(\mathbf{q})\zeta(\mathbf{q}+2\mathbf{k}_\mathrm{L})}{[z_s^2 - z_s\Delta_0 - \bar{\nu}_\mathrm{g}\omega_\mathrm{res}][z_s^2 - z_s\Delta_{-1} - \bar{\nu}_\mathrm{g}\omega_\mathrm{res}] - \tilde{\nu}_\mathrm{g}^2\omega_\mathrm{res}^2} \quad (75)$$

$$I^\mathrm{Ren} = \frac{\bar{N}_\mathrm{e}\gamma_\mathrm{vac}}{2\pi\mathrm{i}}\left\{2 + \ln\left(\frac{\Lambda}{\omega_\mathrm{res}}\right)\right\} + \frac{z_s\bar{N}_\mathrm{e}\gamma_\mathrm{vac}c^2}{4\mathrm{i}\pi^2\omega_\mathrm{res}}\int_0^{2k_\mathrm{L}} \mathrm{d}q_z \int \mathrm{d}q_x \mathrm{d}q_y \frac{[z_s^2 - z_s\Delta_{-1} - \bar{\nu}_\mathrm{g}\omega_\mathrm{res}]}{|\mathbf{q}|\left\{[z_s^2 - z_s\Delta_0 - \bar{\nu}_\mathrm{g}\omega_\mathrm{res}][z_s^2 - z_s\Delta_{-1} - \bar{\nu}_\mathrm{g}\omega_\mathrm{res}] - \tilde{\nu}_\mathrm{g}^2\omega_\mathrm{res}^2\right\}}$$
$$+ \frac{z_s\tilde{N}_\mathrm{e}\tilde{\nu}_\mathrm{g}\gamma_\mathrm{vac}c^3}{2\mathrm{i}\pi^2}\int_0^{2k_\mathrm{L}} \mathrm{d}q_z \int \mathrm{d}q_x \mathrm{d}q_y \frac{|\mathbf{q}|^{-1/4}|\mathbf{q}+2\mathbf{k}_\mathrm{L}|^{-1/4}}{[z_s^2 - z_s\Delta_0 - \bar{\nu}_\mathrm{g}\omega_\mathrm{res}][z_s^2 - z_s\Delta_{-1} - \bar{\nu}_\mathrm{g}\omega_\mathrm{res}] - \tilde{\nu}_\mathrm{g}^2\omega_\mathrm{res}^2} \quad (78)$$

V_-. The first integral in equation (73) represents the contribution from the higher polaritonic energy bands where the polaritons can be considered as free photons. Apart from the restriction $|q_z| > 2k_\mathrm{L}$, which essentially means that the photons are far off-resonant, it has the same form as the integral

$$I_0 = \frac{\bar{N}_\mathrm{e}}{\mathrm{i}\hbar^2}\int_{-\infty}^{\infty} \mathrm{d}q_x \mathrm{d}q_y \mathrm{d}q_z \frac{\zeta(\mathbf{q})^2}{z_s - c|\mathbf{q}| + \omega_\mathrm{res}} \quad (74)$$

which appears in the calculation of the free-space SE rate. The second contribution arises from the two lowest energy bands for negative quasi-momentum $0 > q_z > -k_\mathrm{L}$, and the third integral represents the corresponding contribution for positive quasi momentum q_z. It is not hard to see that equation (73) reduces to the free integral (74) in absence of a ground-state BEC, *i.e.*, for $\bar{\nu}_\mathrm{g} = \tilde{\nu}_\mathrm{g} = 0$ and to equation (49) if the BEC is homogeneous (for $\tilde{\nu}_\mathrm{g} = 0$ but $\bar{\nu}_\mathrm{g} \neq 0$).

We now return to the evaluation of equation (73). Shifting the integration variable q_z and exploiting the symmetries $\zeta(-\mathbf{k}) = \zeta(\mathbf{k})$ and $\Delta_1(-\mathbf{q}) = \Delta_{-1}(\mathbf{q})$ allows us to combine the last two integrals into a more convenient form. To simplify the process of renormalisation it is also advantageous to calculate $I - I_0$ instead of I alone since in this difference the divergence appearing in I_0 is canceled. We then find

see equation (75) above.

To calculate these expressions we have to make one further approximation by neglecting the angular dependence in $\zeta(\mathbf{q}) = \sin(\theta)\omega_\mathrm{res}|\mathbf{d}|[\hbar/(2(2\pi)^3\varepsilon_0 c|\mathbf{q}|]^{1/2}$, where θ is the angle between $\mathbf{q}$ and the atomic dipole moment $\mathbf{d}$. Replacing $\sin(\theta)$ by $\sqrt{2/3}$ for all values of $\mathbf{q}$ leads to the correct result in absence of a BEC and should produce qualitatively correct results for the case under consideration. Doing this approximation in the case of a homogeneous BEC just amounts in replacing a factor of 4/5 by 2/3 in the modifications of the SE rate (33). We remark that neglecting the dependence of ζ on $\sin(\theta)$ does only symmetrise the integrand in the (q_x, q_y) plane. This does not correspond to an isotropic band model because the asymmetry between q_z and (q_x, q_y) still persists.

The calculation of the first integral of equation (75), which roughly corresponds to the contribution of $-I_0$, is quickly done and results in

$$\frac{2\mathrm{i}\bar{N}_\mathrm{e}}{\hbar^2}\int_0^{2k_\mathrm{L}} \mathrm{d}q_z \int_{-\infty}^{\infty} \mathrm{d}q_x \mathrm{d}q_y \frac{\zeta(\mathbf{q})^2}{z_s - c|\mathbf{q}| + \omega_\mathrm{res}} = \frac{\bar{N}_\mathrm{e}\gamma_\mathrm{vac}}{2\pi\mathrm{i}}\left\{2 + 2\ln\left(\frac{\Lambda}{\omega_\mathrm{res}}\right) + \mathrm{i}\pi\mathrm{sgn}(\mathrm{Im}(z_s))\right\}. \quad (76)$$

Here Λ is a cut-off which usually is taken to be $\Lambda = m_\mathrm{e}c^2/\hbar$, where m_e is the electron's mass (see, *e.g.*, Ref. [12]).

The dependence on the sign of $\mathrm{Im}(z_s)$ originally comes from a logarithm of the form $\ln(-(\omega_\mathrm{res} + z_s)/\Lambda)$. This expression can be reduced to the one presented in equation (76) by doing a generalized Wigner-Weisskopf approximation as it was introduced above in the case of a running laser wave. Though $\mathrm{Im}(z_s)$ is also much smaller than any other quantity in the above logarithm, it determines the sign of the imaginary part of the logarithm's argument. Since the logarithm has a branch cut along the negative real axis, this sign determines on which side of the cut we are.

It is also worth remarking that equation (76) is logarithmically divergent with Λ although we did not subtract the free-electron part, a step which in the free-space calculation is done to remove a linearly divergent contribution (see, *e.g.*, Ref. [12]). This is because the integration over q_z does not extend to infinity, thus reducing the degree of divergence by one.

As has been already mentioned, equation (76) very roughly corresponds to the negative of the free-space integral I_0. As a consequence, its contribution will be mostly canceled after the renormalisation of I. This renormalisation is easily done by noting that $I - I_0 = I^\mathrm{Ren} - I_0^\mathrm{Ren}$ so that $I^\mathrm{Ren} = (I - I_0) + I_0^\mathrm{Ren}$, where the superscript "Ren" denotes the renormalized integrals and the renormalized free-space integral is approximately given by

$$I_0^\mathrm{Ren} = \mathrm{i}\frac{\gamma_\mathrm{vac}\bar{N}_\mathrm{e}}{2\pi}\left\{\ln\left(\frac{\Lambda}{\omega_\mathrm{res}}\right) + \mathrm{i}\pi\mathrm{sgn}(\mathrm{Im}(z_s))\right\}. \quad (77)$$

This allows us to derive from equation (75) the expression

see equation (78) above.

To reduce equation (78) it is convenient to introduce the scaled variables of integration $u := q_z/(2k_\mathrm{L})$ and

$$I^{\mathrm{Ren}} = \frac{\bar{N}_{\mathrm{e}}\gamma_{\mathrm{vac}}}{2\pi\mathrm{i}}\left\{2+\ln\left(\frac{\Lambda}{\omega_{\mathrm{res}}}\right)\right\} + \frac{\gamma_{\mathrm{vac}}\bar{N}_{\mathrm{e}}}{2\pi\mathrm{i}}\int_0^1 \mathrm{d}u\int_0^\infty \mathrm{d}v \frac{(ck_{\mathrm{L}}/\omega_{\mathrm{res}})[f_0(z_s)-\sqrt{(u-1)^2+v}]}{\sqrt{u^2+v}\{[f_0(z_s)-\sqrt{u^2+v}][f_0(z_s)-\sqrt{(u-1)^2+v}]-f_1(z_s)\}}$$
$$+\frac{\gamma_{\mathrm{vac}}\tilde{N}_{\mathrm{e}}}{4\pi\mathrm{i}}\frac{\tilde{\nu}_{\mathrm{g}}}{z_s}\int_0^1 \mathrm{d}u\int_0^\infty \mathrm{d}v \frac{(u^2+v)^{-1/4}((u-1)^2+v)^{-1/4}}{[f_0(z_s)-\sqrt{u^2+v}][f_0(z_s)-\sqrt{(u-1)^2+v}]-f_1(z_s)} \quad (79)$$

$v := (q_x^2+q_y^2)/(4k_{\mathrm{L}}^2)$ and the abbreviations (35, 36) for the evaluation. Equation (78) then becomes

see equation (79) above.

This expression can be further reduced by switching to the integration variable $v' = \sqrt{(u-1)^2+v}$ and exchanging the sequence of integration so that one first integrates over u. This allows us to reduce the integral I to a number of one-dimensional integrals and leads us to our final analytical result (34).

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